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The constant source problem with Fresnel reflection

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Abstract
The classic constant source problem for a half-space is generalized to include the effect of refraction at the boundary by inclusion of the Fresnel boundary conditions. The problem is solved using the Wiener–Hopf technique with both specular and diffuse reflection. The non-singular Fredholm integral equations that arise for the surface angular distribution are solved numerically and the solutions are illustrated by a number of results in graphical and tabular forms. The significant effect of refraction on the surface flux and current and the associated angular distributions is highlighted.

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1. Introduction

There are three classic half-space problems in particle transport theory, as described by the Boltzmann equation: (a) the Milne problem, (b) the albedo problem and (c) the constant source problem. The oldest of these problems, that due to Milne, was posed and approximately solved by Milne in the 1920s in connection with the flow of energy inside stars. The albedo problem also had extra-terrestrial origins in the need to calculate the reflection of sunlight from planets. The third problem was introduced by nuclear reactor theorists although they also borrowed the Milne and albedo problems. The major difficulty in obtaining analytical solutions to all three problems arises from the boundary condition; namely a known incoming flux onto the surface of the half-space but an unknown emergent flux of particles (neutrons or photons). This leads to the associated mathematical problem being one involving mixed boundary conditions. These problems have been solved using a variety of methods but the earliest was that of Wiener and Hopf. A detailed examination of these problems can be found in Davison (1957) and using a different approach in Case and Zweifel (1967).

In two recent papers (Williams 2005, 2006), we have examined the effect of Fresnel boundary conditions on the behaviour of the radiation flux in the Milne problem and the

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albedo problem. In the case of the Milne problem it is shown that the extrapolated endpoint, the emergent angular flux and the spatial distribution were strongly dependent on the value of the refractive index. Analogous conclusions were also found for the albedo problem. It was also shown that the Wiener–Hopf method (Davison 1957, Williams 1971) provides a useful technique for casting the problem into a mathematical form that is suitable for numerical evaluation. We apply the same technique here to the classical constant source problem in which a uniformly distributed source is present in the half-space. The internally reflected and transmitted angular distributions are calculated, along with the surface flux and current for specular and diffuse reflection at the boundary. In general, the work described here is relevant to infra-red radiation and is a contribution to the understanding of the transmission of radiation in the tissue of the neo-natal head. It also has applications in light transmission through an interface between media with different refractive indices. The present work also completes the solutions of these three classic problems in the optical region where refraction is important.

2. Definition of the problem

We assume one-speed transport theory throughout which is valid for radiation in the infra-red region. Let $\phi(x, \mu)$ be the radiative flux defined by the following transport equation (Davison 1957, Chandrasekhar 1960).

$$\mu\frac{\partial\phi(x,\mu)}{\partial x} + \Sigma\phi(x,\mu) = \frac{\Sigma_s}{2\pi}\int \mathrm{d}\mathbf{\Omega}' p(\mu_0)\phi(x,\mu') + \frac{Q_0}{2}. \tag{1}$$

In equation (1), $\Sigma = \Sigma_s + \Sigma_a$, where Σ_s is the macroscopic scattering cross section and Σ_a is the macroscopic absorption cross section. $p(\mu_0)$ is the phase function which determines the change in direction during a scattering event. $\mu_0 = \mathbf{\Omega}\cdot\mathbf{\Omega}'$, the cosine of the angle of scattering. Q_0 is the source strength.

In order to simplify our analysis, we assume that the phase function can be written as

$$p(\mu_0) = \tfrac{1}{2}(1-\bar{\mu}_0) + \bar{\mu}_0\delta(\mu_0 - 1), \tag{2}$$

where $\bar{\mu}_0$ is the mean cosine of scattering. Equation (2) is the so-called transport approximation (Davison 1957). Inserting (2) into (1) and re-formulating, we find

$$\mu\frac{\partial\phi(x,\mu)}{\partial x} + \phi(x,\mu) = \frac{c}{2}\int_{-1}^{1}\mathrm{d}\mu'\phi(x,\mu') + \frac{1}{2}, \tag{3}$$

where

$$c = \frac{\Sigma_s(1-\bar{\mu}_0)}{\Sigma_s(1-\bar{\mu}_0) + \Sigma_a} \tag{4}$$

and x is scaled by the transport mean free path $1/\Sigma_{\mathrm{tr}}$, where $\Sigma_{\mathrm{tr}} = \Sigma_s(1-\bar{\mu}_0) + \Sigma_a$. Also we have used a unit source.

If we are to consider the problem with Fresnel boundary conditions (Born and Wolf 1999), then we must write for $\mu > 0$,

(a) specular reflection

$$\phi(0,\mu) = R(\mu)\phi(0,-\mu) \tag{5}$$

and

(b) diffuse reflection

$$\phi(0,\mu) = R_d(\mu)J_0, \tag{6}$$

where

$$J_0 = 2\int_0^1 \mathrm{d}\mu\, \mu\phi(0, -\mu). \tag{7}$$

In equation (5), $R(\mu)$ is the Fresnel reflection coefficient arising from the internal reflection of photons at the interface and given by Born and Wolf (1999) and Aronson (1995) as

$$\begin{aligned} R(\mu) &= \frac{1}{2}\left[\left(\frac{\mu - n\mu_t}{\mu + n\mu_t}\right)^2 + \left(\frac{\mu_t - n\mu}{\mu_t + n\mu}\right)^2\right] \qquad \mu_c \leqslant \mu \leqslant 1 \\ &= 1 \qquad 0 \leqslant \mu \leqslant \mu_c \end{aligned} \tag{8}$$

with $\mu_t^2 = 1 - n^2 + n^2\mu^2$ (μ is the cosine of the angle of refraction). The critical angle for internal reflection is given by

$$\mu_c = \frac{\sqrt{n^2 - 1}}{n} \tag{9}$$

or $\vartheta_c = \sin^{-1}(1/n)$, with n being the refractive index of the medium in $x > 0$. We note from equation (8) that if $n = 1$, $\mu_c = 0$ and $R(\mu) = 0$, i.e. all photons are transmitted as in the classical case. In the present case of a medium–vacuum interface and infra-red photons, the boundary condition is far more complicated and consequently the problem becomes much richer in content.

In the case of diffuse reflection, $R_d(\mu)$ will depend on the nature of the surface, but simple roughness generally assumes the Lambert law in which $R_d(\mu)$ is independent of angle (Modest 2003).

Equation (8) shows that total internal reflection occurs for photons in the range $0 \leqslant \mu \leqslant \mu_c$. This is for photons in the medium. It is clear that photons on the vacuum side of the interface are refracted as they pass through the interface. Because $n > 1$, the refracted ray is bent towards the normal. Thus as the vacuum angle varies between 0 and $\pi/2$, the direction of the refracted ray varies between 0 and ϑ_c. On the vacuum side, because the photons are moving from a less dense to a denser medium, the reflection coefficient changes in form (Sobolev 1963) and the transmission factor for emerging photons is $1 - \tilde{R}(\mu)$, where

$$\tilde{R}(\mu) = \frac{1}{2}\left[\left(\frac{n\mu - \tilde{\mu}}{n\mu + \tilde{\mu}}\right)^2 + \left(\frac{\mu - n\tilde{\mu}}{\mu + n\tilde{\mu}}\right)^2\right] \qquad 0 \leqslant \mu \leqslant 1 \tag{10}$$

and

$$\tilde{\mu} = \left[1 - \frac{1 - \mu^2}{n^2}\right]^{1/2}. \tag{11}$$

For the constant source problem equation (10) is not required, but we give it for completeness and also because it is referred to in the appendix.

3. Solution of the constant source problem

We define the Laplace transform of the angular flux as

$$\bar{\phi}(s, \mu) = \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-sx}\phi(x, \mu). \tag{12}$$

Apply the transform to equation (3), divide by $(1 + s\mu)$ and integrate over $\mu(-1, 1)$ to get

$$\int_{-1}^0 \frac{\mathrm{d}\mu\, \mu\phi(0, \mu)}{1 + s\mu} + \int_0^1 \frac{\mathrm{d}\mu\, \mu\phi(0, \mu)}{1 + s\mu} = (1 - cK(s))\bar{\phi}_0(s) - \frac{1}{s}K(s), \tag{13}$$

where

$$K(s) = \frac{1}{2s}\log\left(\frac{1+s}{1-s}\right) \tag{14}$$

and

$$\bar{\phi}_0(s) = \int_{-1}^{1} \mathrm{d}\mu\, \bar{\phi}(s,\mu). \tag{15}$$

We define

$$g_-(s) = \int_{-1}^{0} \frac{\mathrm{d}\mu\, \mu\phi(0,\mu)}{1+s\mu} \tag{16}$$

$$g_+(s) = \int_{0}^{1} \frac{\mathrm{d}\mu\, \mu\phi(0,\mu)}{1+s\mu} \tag{17}$$

whence (13) becomes

$$1 + csg_-(s) + csg_+(s) = (1 - cK(s))\bar{\Psi}(s), \tag{18}$$

where $\bar{\Psi}(s) = cs\bar{\phi}_0(s) + 1$. Using (5) for specular reflection or (6) for diffuse reflection we find

$$g_+(s) = \int_{0}^{1} \frac{\mathrm{d}\mu\, \mu\Phi(\mu)}{1+s\mu}. \tag{19}$$

For specular reflection

$$\Phi(\mu) = R(\mu)\phi(0,-\mu) \tag{20}$$

and for diffuse reflection

$$\Phi(\mu) = R_d(\mu)J_0. \tag{21}$$

We now follow the well-established Wiener–Hopf procedure (Williams 1971) whereby we define the function

$$\tau(s) = \frac{(1-cK(s))(s^2-1)}{s^2-\nu^2} = \frac{\tau_+(s)}{\tau_-(s)}. \tag{22}$$

The quantities $\pm\nu$ are the roots of $1 - cK(s) = 0$. Thus $\tau(s)$ has no zeros in the range $s(-1,1)$ and tends to unity as $|s| \to \infty$. The functions $\tau_\pm(s)$ are defined as

$$\log\tau_\pm(s) = \frac{1}{2\pi\mathrm{i}}\int_{\pm\eta-\mathrm{i}\infty}^{\pm\eta+\mathrm{i}\infty} \frac{\log\tau(u)}{u-s}\,\mathrm{d}u \tag{23}$$

$\tau_+(s)$ is analytic for $\mathrm{Re}(s) < \eta$ and $\tau_-(s)$ for $\mathrm{Re}(s) > -\eta$ where $\eta < 1$. We also note that $g_-(s)$ is analytic for $\mathrm{Re}(s) < 1$. $\bar{\phi}_0(s)$ is analytic for $\mathrm{Re}(s) > -\nu$. Inserting equation (22) into equation (18) and rearranging terms leads to

$$\frac{(s-1)}{\tau_+(s)}\left\{1 + csg_-(s) + cs\int_0^1 \frac{\mathrm{d}\mu\,\mu\Phi(\mu)}{1+s\mu}\right\} = \frac{s^2-\nu^2}{s+1}\frac{1}{\tau_-(s)}\bar{\Psi}(s). \tag{24}$$

Here we see that the right-hand side is analytic in $\mathrm{Re}(s) > -\nu$. The first term in the curly bracket on the left-hand side is analytic in $\mathrm{Re}(s) < \eta$. Unfortunately, the second term is analytic only in the strip $-1 < \mathrm{Re}(s) < \eta$. Thus before we can use the Wiener–Hopf factorization, it is necessary to decompose the second term as follows. Following Cauchy's principle, let us define

$$F(s) = \frac{1}{\tau_+(s)}\int_0^1 \frac{\mathrm{d}\mu\,\mu\Phi(\mu)}{1+s\mu} = F_+(s) - F_-(s), \tag{25}$$

where

$$F_{\pm}(s) = \frac{1}{2\pi \mathrm{i}} \int_{\pm\eta-\mathrm{i}\infty}^{\pm\eta+\mathrm{i}\infty} \frac{\mathrm{d}u}{u-s} \frac{1}{\tau_+(u)} \int_0^1 \frac{\mathrm{d}\mu\, \mu \Phi(\mu)}{1+u\mu} \tag{26}$$

and F_+ and F_- have the same regions of analyticity as τ_+ and τ_-, respectively.

Equation (24) may now be written as

$$\frac{(s-1)}{\tau_+(s)}[csg_-(s)+1] + cs(s-1)F_+(s) = cs(s-1)F_-(s) + \frac{s^2-\nu^2}{s+1}\frac{1}{\tau_-(s)}\bar{\Psi}(s). \tag{27}$$

Now the left-hand side of (27) is analytic in $\mathrm{Re}(s) < \eta$ and the right-hand side in $\mathrm{Re}(s) > -\nu$. Hence we have a common strip of analyticity and the necessary Wiener–Hopf conditions are satisfied. The limit of each side of equation (27) as $|s| \to \infty$ goes linearly as s and so according to the extended form of Liouville's theorem (Titchmarsh 1937), we can write

$$cs(s-1)F_-(s) + \frac{s^2-\nu^2}{s+1}\frac{1}{\tau_-(s)}\bar{\Psi}(s) = C_0 + C_1 s \tag{28}$$

and

$$\frac{(s-1)}{\tau_+(s)}[csg_-(s)+1] + cs(s-1)F_+(s) = C_0 + C_1 s, \tag{29}$$

where C_0 and C_1 are constants to be determined.

To calculate $F_-(s)$, we refer the reader to Williams (1971) whence

$$F_-(s) = -\int_0^1 \frac{\mathrm{d}\mu\, \mu\tau_-(1/\mu)}{1+s\mu}\Phi(\mu). \tag{30}$$

The Laplace transform of the scalar flux can be written from (28) as

$$\bar{\Psi}(s) = \frac{s+1}{s^2-\nu^2}\tau_-(s)\,[C_0 + C_1 s - cs(s-1)F_-(s)] \tag{31}$$

which means from the definition of $\bar{\Psi}(s)$ that

$$\bar{\phi}_0(s) = \frac{1}{cs}\left\{\frac{s+1}{s^2-\nu^2}\tau_-(s)[C_0 + C_1 s - cs(s-1)F_-(s)] - 1\right\} \tag{32}$$

to get C_0 we set $s = 0$ in equation (29), whence

$$C_0 = -\tau_-(0) = -\frac{\nu}{\sqrt{1-c}}. \tag{33}$$

But we know physically that as $x \to \infty$, $\phi_0(x) \sim \text{constant} + e^{-\nu x}$, thus we must choose C_0 and C_1 to remove the pole in (32) at $s = \nu$. To do this we use (33) and

$$C_1 = \frac{\nu}{(1-c)\tau_-(0)} + c(\nu-1)F_-(\nu) = \frac{1}{\sqrt{1-c}} + c(\nu-1)F_-(\nu) \tag{34}$$

and we have used the relation $\tau_-(-s) = 1/\tau_+(s)$ from the definitions in equation (23). Inserting C_0 and C_1 into (32), we find

$$\bar{\Psi}(s) = \frac{(s+1)}{(s+\nu)}\tau_-(s)\left\{\frac{1}{\sqrt{1-c}} + cs\int_0^1 \mathrm{d}\mu' \frac{\mu'(1+\mu')\tau_-(1/\mu')\Phi(\mu')}{(1+\nu\mu')(1+s\mu')}\right\}. \tag{35}$$

To regain the flux itself we can return to the definition of $\bar{\Psi}(s)$, but in order to calculate the surface angular distribution we only need to integrate equation (3) to get

$$\phi(0,-\mu) = \frac{1}{2}\bar{\Psi}\left(\frac{1}{\mu}\right) \tag{36}$$

whence from (35), setting $s = 1/\mu$

$$\phi(0,-\mu) = \frac{1}{2}H(\mu)\left\{\frac{1}{\sqrt{1-c}} + c\int_0^1 \frac{\mathrm{d}\mu'\,\mu'\Phi(\mu')H(\mu')}{\mu+\mu'}\right\}, \tag{37}$$

where we have defined Chandrasekhar's H function by

$$H(\mu) = \frac{(1+\mu)\tau_-(1/\mu)}{1+\nu\mu}. \tag{38}$$

There are two integral quantities of some interest, namely the surface scalar flux and the current. The surface flux can be obtained readily from the Laplace transform by using the relation

$$\lim_{s\to\infty} s\bar{\phi}_0(s) = \phi_0(0) \tag{39}$$

whence

$$\phi_0(0) = \frac{1}{c}\left[\frac{1}{\sqrt{1-c}} - 1\right] + \int_0^1 \mathrm{d}\mu\, H(\mu)\Phi(\mu) \tag{40}$$

and the current becomes

$$J(0) = \int_{-1}^1 \mathrm{d}\mu\,\mu\phi(0,\mu) = -\frac{h_1}{2\sqrt{1-c}} + \sqrt{1-c}\int_0^1 \mathrm{d}\mu\,\mu H(\mu)\Phi(\mu), \tag{41}$$

where h_1 is the first moment of the H function (Chandrasekhar 1960). All of these results depend on the form taken by $\Phi(\mu)$ as defined by equations (20) and (21).

If the spatial variation of the scalar flux is required, then it is necessary to perform the inverse Laplace transform of $\bar{\phi}_0(s)$. This is a tedious but straightforward procedure, details of which may be found in Williams (1971). The result is of the form

$$\begin{aligned}\phi_0(x) = \frac{1}{1-c} &- \frac{(1-\nu^2)}{c(c+\nu^2-1)H(1/\nu)}\left\{\frac{1}{\sqrt{1-c}} - c\nu\int_0^1 \mathrm{d}\mu\frac{\mu H(\mu)\Phi(\mu)}{1-\nu\mu}\right\}\mathrm{e}^{-\nu x}\\ &-\frac{1}{2}\int_0^1 \mathrm{d}\omega\,\mathrm{e}^{-x/\omega}\frac{g(c,\omega)}{H(\omega)}\left\{\frac{1}{\sqrt{1-c}} - cP.\int_0^1 \mathrm{d}\mu\frac{\mu H(\mu)\Phi(\mu)}{\omega-\mu}\right\},\end{aligned} \tag{40a}$$

where P. denotes principal value and

$$\frac{1}{g(c,\omega)} = \left(1 - \frac{c\omega}{2}\log\left(\frac{1+\omega}{1-\omega}\right)\right)^2 + \left(\frac{c\pi\omega}{2}\right)^2.$$

Equation (40a) is physically interesting as it shows the structure of the solution. Namely, that for very large distances from the surface the scalar flux goes over to the infinite medium value. For distances a few mean free paths from the surface the integral transient term is negligible and the solution has diffusion-like properties (see section 4). Near the surface, diffusion theory fails and the complete solution is needed. For $x = 0$ we have the special case given by equation (40). The value of $\phi(0,-m)$ is given by a solution of the integral equation (42) for specular reflection or the expression (43) below for diffuse reflection. For diffuse reflection, the result is explicit.

3.1. Specular reflection

In this case we use $\Phi(\mu) = R(\mu)\phi(0,-\mu)$ in equation (37) which then leads to a Fredholm integral equation of the form

$$\phi(0,-\mu) = \frac{1}{2}H(\mu)\left\{\frac{1}{\sqrt{1-c}} + c\int_0^1 \frac{\mathrm{d}\mu'\,\mu'R(\mu')\phi(0,-\mu')H(\mu')}{\mu+\mu'}\right\}. \tag{42}$$

This has to be solved numerically and the results inserted into (40) and (41). It is useful to note, however, that if we set $R(\mu) = 1$ in equation (42), i.e. perfect specular reflection, and use the properties of the H-functions, we find that $\phi(0, -\mu) = 1/2(1-c)$. This is the infinite medium solution as we expect.

3.2. *Diffuse reflection*

In the case of diffuse reflection, the integral equation becomes

$$\phi(0, -\mu) = \frac{1}{2}H(\mu)\left\{\frac{1}{\sqrt{1-c}} + cJ_0\int_0^1 \frac{\mathrm{d}\mu'\,\mu' R_d(\mu')H(\mu')}{\mu+\mu'}\right\}. \tag{43}$$

To get J_0 we multiply equation (43) by μ and integrate over $\mu(0, 1)$. Then using the properties of then H functions (Chandrasekhar 1960), we get

$$\frac{1}{J_0} = \frac{\sqrt{1-c}}{h_1}\left(1 - 2\int_0^1 \mathrm{d}\mu\,\mu R_d(\mu)[1-\sqrt{1-c}H(\mu)]\right). \tag{44}$$

Thus we find

$$\phi_0(0) = \frac{1}{c}\left[\frac{1}{\sqrt{1-c}} - 1\right] + J_0\int_0^1 \mathrm{d}\mu\,H(\mu)R_d(\mu) \tag{45}$$

and

$$J(0) = \int_{-1}^1 \mathrm{d}\mu\,\mu\phi(0,\mu) = -\frac{h_1}{2\sqrt{1-c}} + \sqrt{1-c}J_0\int_0^1 \mathrm{d}\mu\,\mu H(\mu)R_d(\mu). \tag{46}$$

If the diffuse reflection coefficient is independent of μ, as might be the case for a rough surface, we find

$$\phi_0(0) = \frac{1}{c}\left[\frac{1}{\sqrt{1-c}} - 1\right]\left[\frac{1 - R_d + 2R_d h_1(1+\sqrt{1-c})}{1 - R_d + 2R_d h_1\sqrt{1-c}}\right] \tag{47}$$

and

$$J(0) = \frac{h_1(1-R_d)}{2\sqrt{1-c}(1 - R_d + 2R_d h_1\sqrt{1-c})}. \tag{48}$$

Note that when $R_d = 1$, $J(0) = 0$, because all particles are reflected and there is no leakage. Also $\phi_0(0) = 1/(1-c)$ which is the infinite medium solution, as expected.

We note that the emergent angular distribution from the half space is

$$\phi_{\mathrm{out}}(\mu) = \frac{1}{n^2}(1 - R(\mu))\phi(0, -\mu), \tag{49}$$

where

$$\begin{aligned} \mu &= \sqrt{1 - \frac{1-\mu_t^2}{n^2}} && 0 < \mu_t < 1 \\ \mu_t &= \sqrt{1 - n^2(1-\mu^2)} && \mu_c < \mu < 1. \end{aligned} \tag{50}$$

4. Diffusion theory

A useful and effective alternative to transport theory can be the diffusion approximation. It is reasonably accurate for small values of $1 - c$ and for positions a few mean free paths from boundaries. In this section we show how a diffusion solution can be obtained.

The diffusion equation for this problem can be written as (Davison 1957)

$$\frac{1}{3}\frac{\mathrm{d}^2\phi_0(x)}{\mathrm{d}x^2} - (1-c)\phi_0(x) + 1 = 0. \tag{51}$$

The associated boundary condition at a free surface with Fresnel reflection has been given in Williams (2005) and takes the form

$$\phi_0(0) = \lambda_s \phi_0'(0), \tag{52}$$

where for specular reflection

$$\lambda_s = \frac{2}{3}\left(\frac{1+3R_2}{1-2R_1}\right) \tag{53}$$

and

$$R_j = \int_0^1 \mathrm{d}\mu\, \mu^j R(\mu). \tag{54}$$

For diffuse reflection,

$$\phi_0(0) = \lambda_d \phi_0'(0), \tag{55}$$

where

$$\lambda_d = \frac{2}{3}\left(\frac{1+2R_1}{1-2R_1}\right). \tag{56}$$

Equations (53) and (56) show that the extrapolation distances for specular and diffuse reflection are λ_s and λ_d, respectively and, moreover, that they are independent absorption, i.e. the value of c.

Solving equation (51) subject to the boundary conditions we find

$$\phi_0(x) = \frac{1}{1-c}\left[1 - \frac{\mathrm{e}^{-\kappa x}}{1+\lambda\kappa}\right] \tag{57}$$

with $\kappa^2 = 3(1-c)$. λ takes the appropriate value according to whether we have specular or diffuse reflection. Thus we find

$$\phi_0(0) = \frac{\lambda\kappa}{(1-c)(1+\lambda\kappa)}, \qquad J(0) = -\frac{\kappa}{3(1-c)(1+\lambda\kappa)}. \tag{58}$$

5. Numerical results

In this section we will present some numerical results based upon the solution of the integral equations for specular reflection only, i.e. equation (42). In table 1 we show the surface flux and current for $n = 4/3$ and a range of c values. The results are for transport and diffusion theories. It is clear that the surface currents are close, which means that diffusion theory is remarkably accurate. This is probably because of conservation requirements. The accuracy of diffusion theory is not nearly as good for the surface flux, the error in which varies from 24% at $c = 0.1$ to 11% at $c = 0.99$.

Tables 2 and 3 give results for the surface flux and current for $c = 0.9$ and $c = 0.99$, respectively. Once again we find that the surface currents for transport and diffusion theories are very close, to within 1.0% for the $c = 0.9$ and 0.99 cases. The diffusion surface flux is less accurate but the error is less than 12%.

Finally, in order to illustrate how Fresnel reflection and Snell refraction affect the emergent angular distribution from the surface of the half-space, we show figure 1. This gives the classic result for no reflection, i.e. $n = 1$, as a comparison. The angle θ is the angle away from the

Table 1. Surface flux and current for transport and diffusion theories, $n = 4/3$.

c	$\phi_0(0)^T$	$\phi_0(0)^D$	$-J(0)^T$	$-J(0)^D$
0.1	0.9263	0.6981	0.1446	0.1377
0.2	1.0278	0.7853	0.1611	0.1549
0.3	1.1554	0.8967	0.1818	0.1768
0.4	1.3206	1.0415	0.2088	0.2054
0.5	1.5437	1.2361	0.2455	0.2438
0.6	1.8633	1.5132	0.2985	0.2984
0.7	2.3628	1.9447	0.3821	0.3836
0.8	3.2693	2.7280	0.5356	0.5380
0.9	5.5408	4.7035	0.9270	0.9276
0.91	5.9852	5.0924	1.0044	1.0043
0.92	6.5183	5.5599	1.0974	1.0965
0.93	7.1719	6.1344	1.2119	1.2098
0.94	7.9962	6.8608	1.3566	1.3531
0.95	9.0755	7.8151	1.5469	1.5413
0.96	10.566	9.1375	1.8105	1.8021
0.97	12.795	11.125	2.2066	2.1941
0.98	16.621	14.556	2.8898	2.8708
0.99	25.493	22.575	4.4835	4.4522

Table 2. Surface flux and current for transport and diffusion theories, $c = 0.9$.

n	$\phi_0(0)^T$	$\phi_0(0)^D$	$-J(0)^T$	$-J(0)^D$
1.0	2.4025	2.5942	1.3049	1.2971
1.1	3.8179	3.2112	1.1956	1.1890
1.2	4.6613	3.8746	1.0754	1.0728
1.3	5.3415	4.5051	0.9623	0.9624
1.4	5.9077	5.0807	0.8601	0.8616
1.5	6.3877	5.5963	0.7690	0.7713
1.6	6.7969	6.0535	0.6886	0.6912
1.7	7.1473	6.4568	0.6180	0.6206
1.8	7.4481	6.8119	0.5561	0.5584
1.9	7.7097	7.1245	0.5016	0.5036
2.0	7.9384	7.3998	0.4535	0.4554

Table 3. Surface flux and current for transport and diffusion theories, $c = 0.99$.

n	$\phi_0(0)^T$	$\phi_0(0)^D$	$-J(0)^T$	$-J(0)^D$
1.0	9.0909	10.314	5.1359	5.1572
1.1	14.903	13.442	4.9882	4.9774
1.2	19.532	17.197	4.7857	4.7615
1.3	24.026	21.209	4.5608	4.5308
1.4	28.400	25.323	4.3265	4.2942
1.5	32.662	29.440	4.0892	4.0574
1.6	36.772	33.493	3.8543	3.8244
1.7	40.702	37.434	3.6257	3.5978
1.8	44.419	41.229	3.4063	3.3795
1.9	47.962	44.857	3.1955	3.1709
2.0	51.328	48.303	2.9940	2.9728

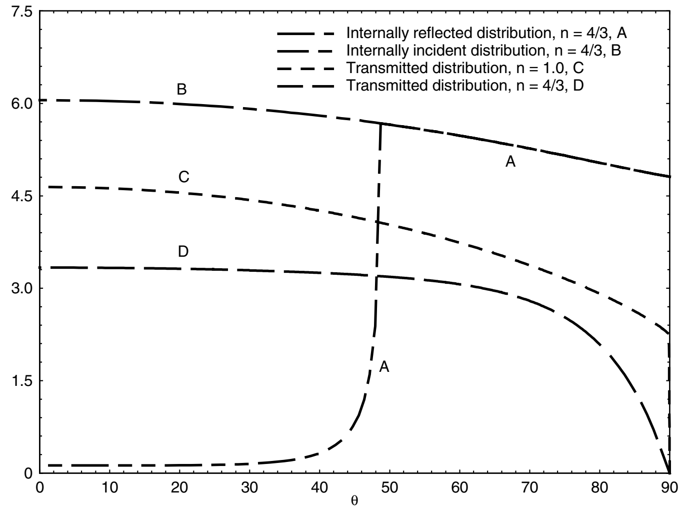

Figure 1. Emergent and reflected angular distributions, $c = 0.95$.

normal to the surface so that $\theta = 0$ is normal incidence and $\theta = 90$ is grazing incidence. Also in the figure is the emergent distribution for $c = 0.95$ and $n = 4/3$. This goes to zero at grazing incidence in contrast to the $n = 1$ case. Additionally there is the internal incident distribution before transmission through the Fresnel surface and the internally reflected distribution which clearly shows the angular cut-off due to total internal reflection.

6. Conclusions and summary

This paper presents the solution of the third classic half-space problem with Fresnel internal reflection; also it gives the case for diffuse reflection which is more appropriate for rough surfaces. The solution follows the general principles laid down in our two previous works (Williams 2005, 2006) in which the Wiener–Hopf method is employed. These seemingly academic problems have more practical value than is often realized as they provide benchmarks for more complex methods which deal with arbitrary geometries and also highlight the important features of the solution. This can be seen in the spatial variation of the scalar flux as given by equation (40*a*) and the emergent angular spectrum given by equations (42) and (43). In the appendix, some additional results are given for the Milne and albedo problems that were omitted from the earlier papers.

Appendix: additional results for the Milne and Albedo problems

In the two earlier publications (Williams 2005, 2006), for the Milne and albedo problems, we did not give the surface flux and current. This is readily done by using equation (39) for the results in those papers and by direct integration.

Albedo case—specular

$$\phi_0(0) = \frac{\mu_0}{\bar{\mu}_0}(1 - \tilde{R}(\mu_0))H(\bar{\mu}_0) + \int_0^1 \mathrm{d}\mu R(\mu)H(\mu)\phi(0, -\mu) \tag{A.1}$$

$$J(0) = \sqrt{1-c}\mu_0(1-\tilde{R}(\mu_0))H(\bar{\mu}_0) + \sqrt{1-c}\int_0^1 \mathrm{d}\mu\, \mu R(\mu)H(\mu)\phi(0,-\mu), \tag{A.2}$$

where μ_0 is the cosine of the incident direction of the particle and

$$\bar{\mu}_0 = \sqrt{1 - \frac{1-\mu_0^2}{n^2}}. \tag{A.3}$$

Milne case—specular

$$\phi_0(0) = A_0 + \int_0^1 \mathrm{d}\mu R(\mu)\phi(0,-\mu)\frac{(1+\nu\mu)H(\mu)}{1+\mu} \tag{A.4}$$

$$J(0) = -A_0\frac{\sqrt{1-c}}{\nu} - \int_0^1 \mathrm{d}\mu \frac{\mu R(\mu)\phi(0,-\mu)}{1+\mu}\left[\mu - \frac{\sqrt{1-c}}{\nu}(1+\nu\mu)H(\mu)\right]. \tag{A.5}$$

A_0 is an arbitrary constant fixed by the normalization at the surface. In fact the only meaningful quantity for the Milne problem is $J(0)/\phi_0(0)$ which is independent of A_0.

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